ANSWER TO QUERY. (SEE PAGE 128.)

BY PROF. D. J. MC ADAM, WASHINGTON, PENNSYLVANIA.

As preliminary to an answer to Mr. Baker's Query I submit for publication the following problem and its solution by Prof. G. B. Vose, not knowing whether it has ever been published or not.

Problem. — It is required to determine the relation which exists between the fifteen angles which six spheres make with each other.

Solution.—Let x_1, y_1, z_1, r_1 be the coordinates of the center and the radius of the first sphere, with similar expressions for the remaining spheres.

Let (1, 2) be the cosine of the angle of the radii of the spheres 1 and 2, drawn to a common point. The square of the distance between the centers of 1 and 2 may be expressed in two different ways; viz.,

$$r_1^2+r_2^2-2r_1r_2(1,2) \text{ and } (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2.$$

Equating these expressions and transposing, we have

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-r_1^2-r_2^2 = -2r_1r_2(1,2), \qquad (1)$$

Similarly

$$\begin{aligned}
&(x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2-r_1^2-r_3^2 &&= -2r_1r_3(1,3), &&(2)\\
&(x_1-x_4)^2+\&c. &&= -2r_1r_4(1,4), &&(3)\\
&(x_1-x_5)^2+\&c. &&= -2r_1r_5(1,5), &&(4)\\
&(x_1-x_6)^2+\&c. &&= -2r_1r_6(1,6), &&(5)\\
&(x_2-x_3)^2+\&c. &&= -2r_2r_3(2,3), &&(6)\\
&(x_2-x_4)^2+\&c. &&= -2r_2r_4(2,4), &&(7)\\
&(x_2-x_5)^2+\&c. &&= -2r_2r_5(2,5), &&(8)\\
&(x_2-x_6)^2+\&c. &&= -2r_2r_6(2,6), &&(9)\\
&(x_3-x_4)^2+\&c. &&= -2r_3r_4(3,4), &&(10)\\
&(x_3-x_5)^2+\&c. &&= -2r_3r_5(3,5), &&(11)\\
&(x_3-x_6)^2+\&c. &&= -2r_3r_6(3,6), &&(12)\\
&(x_4-x_5)^2+\&c. &&= -2r_4r_5(4,5), &&(13)\\
&(x_4-x_6)^2+\&c. &&= -2r_4r_6(4,6), &&(14)\\
&(x_5-x_6)^2+\&c. &&= -2r_5r_6(5,6). &&(15)
\end{aligned}$$

We must now eliminate the 24 unknowns x_1, y_1, z_1, r_1, x_2, &c., and obtain a relation between (1, 2), (1, 3), (2, 4) &c. This seems at first sight to be a very tedious operation, but the theory of Determinants furnishes an easy method. The Determinant,

$$\begin{vmatrix}
0 & 1 & x_1^2+y_1^2+z_1^2-\ {}_1^2 & x_1 & y_1 & z_1\\
0 & 1 & x_2^2+y_2^2+z_2^2-r_2^2 & x_2 & y_2 & z_2\\
0 & 1 & x_3^2+y_3^2+z_3^2-r_3^2 & x_3 & y_3 & z_3\\
0 & 1 & x_4^2+y_4^2+z_4^2-r_4^2 & x_4 & y_4 & z_4
\end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & x_5^2+y_5^2+z_5^2-r_5^2 & x_5 & y_5 & z_5 \\ 0 & 1 & x_6^2+y_6^2+z_6^2-r_6^2 & x_6 & y_6 & z_6 \end{vmatrix}$$

is evidently zero because the elements of the first column vanish.

For the same reason the determinant

$$\begin{vmatrix} 0 & x_1^2+y_1^2+z_1^2-r_1^2 & 1 & -2x_1 & -2y_1 & -2z_1 \\ 0 & x_2^2+y_2^2+z_2^2-r_2^2 & 1 & -2x_2 & -2y_2 & -2z_2 \\ 0 & x_3^2+y_3^2+z_3^2-r_3^2 & 1 & -2x_3 & -2y_3 & -2z_3 \\ 0 & x_4^2+y_4^2+z_4^2-r_4^2 & 1 & -2x_4 & -2y_4 & -2z_4 \\ 0 & x_5^2+y_5^2+z_5^2-r_5^2 & 1 & -2x_5 & -2y_5 & -2z_5 \\ 0 & x_6^2+y_6^2+z_6^2-r_6^2 & 1 & -2x_6 & -2y_6 & -2z_6 \end{vmatrix}$$

vanishes.

The product of these two Determinants is therefore zero. Hence

$$0=\begin{vmatrix} -2r_1r_1, & -2r_2r_1(2,1), & -2r_3r_1(3,1), & -2r_4r_1(4,1), & -2r_5r_1(5,1), & -2r_6r_1(6,1) \\ -2r_1r_2(1,2), & -2r_2r_2, & -2r_3r_2(3,2), & -2r_4r_2(4,2), & -2r_5r_2(5,2), & -2r_6r_2(6,2) \\ -2r_1r_3(1,3), & -2r_2r_3(2,3), & -2r_3r_3, & -2r_4r_3(4,3), & -2r_5r_3(5,3), & -2r_6r_3(6,3) \\ -2r_1r_4(1,4), & -2r_2r_4(2,4), & -2r_3r_4(3,4), & -2r_4r_4, & -2r_5r_4(5,4), & -2r_6r_4(6,4) \\ -2r_1r_5(1,5), & -2r_2r_5(2,5), & -2r_3r_5(3,5), & -2r_4r_5(4,5), & -2r_5r_5, & -2r_6r_5(6,5) \\ -2r_1r_6(1,6), & -2r_2r_6(2,6), & -2r_3r_6(3,6), & -2r_4r_6(4,6), & -2r_5r_6(5,6), & -2r_6r_6 \end{vmatrix}$$

Now divide each element of the first column by $-2r_1$, each element of the second column by $-2r_2$, &c. Also divide each element of the first rank by r_1, each element of the second rank by r_2, &c., and the Determinant becomes,

$$\begin{vmatrix} 1 & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) \\ (1,2) & 1 & (3,2) & (4,2) & (5,2) & (6,2) \\ (1,3) & (2,3) & 1 & (4,3) & (5,3) & (6,3) \\ (1,4) & (2,4) & (3,4) & 1 & (5,4) & (6,4) \\ (1,5) & (2,5) & (3,5) & (4,5) & 1 & (6,5) \\ (1,6) & (2,6) & (3,6) & (4,6) & (5,6) & 1 \end{vmatrix}=0.$$

This is the solution required. The cosines (2, 1) and (1, 2) are identical but written differently for the sake of symmetry.

To apply this general formula to the question under consideration; since the five spheres are given in position, the angles whose cosines are (1,2) &c., are all known except the five (1, 6), (2, 6), (3, 6), (4, 6) and (5, 6), and the identical angles (6, 1), &c. Since these are to be equal, calling (1,6) &c. x and the known cosines, a, b, &c., our Determinant becomes,

$$\begin{vmatrix} 1 & a & b & c & d & x \\ a & 1 & e & g & l & x \\ b & e & 1 & h & k & x \\ c & g & h & 1 & n & x \\ d & l & k & n & 1 & x \\ x & x & x & x & x & 1 \end{vmatrix}=0.$$

Dividing the determinant by x^2 by dividing the sixth column and sixth rank by x; also taking the lower rank to the top by five transpositions and five changes of sign; also the sixth column to the first by five additions, transpositions and changes of sign, and we get

$$\begin{vmatrix} \frac{1}{x^2} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & b & c & d \\ 1 & a & 1 & e & g & l \\ 1 & b & e & 1 & h & k \\ 1 & c & g & h & 1 & n \\ 1 & d & l & k & n & 1 \end{vmatrix} = 0. \quad \text{Hence } \frac{1}{x^2} \begin{vmatrix} 1 & a & b & c & d \\ a & 1 & e & g & l \\ b & e & 1 & h & k \\ c & g & h & 1 & n \\ d & l & k & n & 1 \end{vmatrix} =$$

$$-\begin{vmatrix} a & b & c & d & 1 \\ 1 & e & g & l & 1 \\ e & 1 & h & k & 1 \\ g & h & 1 & n & 1 \\ l & k & n & 1 & 1 \end{vmatrix} - \begin{vmatrix} b & c & d & 1 & 1 \\ e & g & l & 1 & a \\ 1 & h & k & 1 & b \\ h & 1 & n & 1 & c \\ k & n & 1 & 1 & d \end{vmatrix} - \begin{vmatrix} c & d & 1 & 1 & a \\ g & l & 1 & a & 1 \\ h & k & 1 & b & e \\ 1 & n & 1 & c & g \\ n & 1 & 1 & d & l \end{vmatrix} - \begin{vmatrix} d & 1 & 1 & a & b \\ l & 1 & a & 1 & e \\ k & 1 & b & e & 1 \\ n & 1 & c & g & h \\ 1 & 1 & d & l & k \end{vmatrix} - \begin{vmatrix} 1 & 1 & a & b & c \\ 1 & a & 1 & e & g \\ 1 & b & e & 1 & h \\ 1 & c & g & h & 1 \\ 1 & d & l & k & n \end{vmatrix}$$

From this equation $\frac{1}{x^2}$ is at once found, and hence x and $\cos^{-1}x$.

Having the $\cos^{-1}x$, to construct the sixth sphere. With the centers of the five spheres as vertices and indefinite lines making $\cos^{-1}x$ with the radii of the respective spheres drawn toward the common origin which we have been using, describe circular cones, which will intersect in a common point which is the center of the sixth sphere.

All spheres having this point as center will cut the five spheres at equal angles.

Note on Attraction, by R. J. Adcock, Monmouth, Ill.—If every particle of matter attracts from all directions with an equal constant force, then the attraction between masses or molecules must vary *directly* as their sum and *inversely* as the square of their distance. That no other law is possible follows from the following considerations:—

If every particle attracts with the same constant force, then, that the attraction is as the sum of the masses follows from the axiom that the whole is equal to all its parts. And if the attraction of each particle is a constant force exerted in *all* directions, then, obviously, because the *areas* over which the force is distributed at different distances vary as the squares of the distances, the energy exerted upon a point, or upon a particle of matter at any distance, is inversely as the square of the distance.

Hence, from the known laws of attraction, we have this ultimate proposition:— Assuming the ultimate particles of matter to be infinitely small, every particle attracts, or *draws*, from all directions, with an equal and constant force without regard to the distance of its point of application.